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# AN EFFICIENT PROCEDURE FOR FINDING THE STEADY STATE SOLUTION OF CYCLIC, LINEAR, DISCRETE SYSTEMS

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ABSTRACT

A general numerical method for finding the steady state solution of a cyclic system is presented. The method determines the initial values by enforcing the conditions of periodicity. In this way the initial value is found by integrating through only one cycle, often resulting in a considerable saving of computing effort. The method is applicable to any linear discrete set of difference equations with periodic parameters and forcing functions. The application of the method to a single pole representation of heat flow in buildings is demonstrated.

KEY WORDS Finite difference method Cyclic systems Thermoflow

## INTRODUCTION

Cyclic systems are often encountered in heat transfer and fluid flow. Some examples are cyclic combustion, rotating machines, cyclic flow in cardiovascular circulation, and any system subjected to the diurnal outdoor environment, e.g. buildings. In these systems, the periodicity might arise from periodic boundary conditions, periodic forcing functions, periodic system parameters or combinations of these. Under cyclic conditions a steady state response will eventually arise where the solution of the system also becomes periodic. In many cyclic problems this steady state solution is of interest rather than the initial transient. For instance, in the analysis of heat transfer in buildings no initial conditions for the climatic forcing functions are specified, instead the problem is uniquely determined by the requirement of a periodic solution. In other problems, e.g. the problem of heat transfer in ducts having periodic variations of the cross-sectional area, as treated in Reference 1, it is convenient to neglect the initial region where conditions are not fully developed.

In the solution of initial value problems a simple finite difference method is often used which finds the evolution of the system by a marching algorithm. Although this method is not very accurate, it is simple, fast and requires only modest storage space. However, if the time dependency is cyclic the initial value problem becomes a boundary value problem, and the marching algorithm cannot be used; the system must be solved at once for all time steps. This leads to a finite difference method which requires the solution of a large number of simultaneous equations for accurate results.

Other methods for solving boundary value problems include the popular shooting methods and variational methods[2]. Shooting methods use an arbitrary guess of the initial value, then integrate through a cycle, determine an updated initial value—usually through the secant method—and then integrate through another cycle until convergence is achieved. In case of a linear system the shooting method will converge after integrating through two cycles[2].

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In this paper, a finite difference method is presented which determines the initial period after effectively integrating through just one cycle. The need for the solution of a large system of equations, which the finite difference method (as applied to cyclic systems) requires, is thus circumvented. The method appears to be a generalization of the cyclic tridiagonal matrix algorithm[1].

The method has been successfully applied to thermoflow networks of buildings with time variable parameters. In addition it is applicable to a very wide range of problems in numerical analysis of heat transfer, fluid flow etc. All cyclic phenomena, e.g. reciprocating machines, rotary machines, vortex shedding, flow with periodic boundaries etc. are likely candidates if the equations are linear or can be linearized.

## DESCRIPTION OF THE METHOD

We assume a discrete linear system of the form:

$$\mathbf{x}_{k+1}=\mathbf{A}_k\cdot\mathbf{x}_k+\mathbf{u}_k \tag{1}$$

where $\mathbf{A}_k$ is a square matrix which does not depend on $\mathbf{x}_k$. The vector $\mathbf{u}_k$ is the input to the system which is described by the vector of state variables $\mathbf{x}_k$. The index $k=0,1,2\ldots N$ runs over the periodic independent variable. It is assumed that the system (1) is stable and that the input and system parameters are periodic with respect to the index $k$, with period $N$, and that the steady periodic solution is required. Thus $\mathbf{A}_0=\mathbf{A}_N$, $\mathbf{u}_0=\mathbf{u}_N$ and $\mathbf{x}_0=\mathbf{x}_N$. Note that a system which is period in more than one independent variable can fit into this scheme with $N$ given by the least common multiple of the periods of the independent variables. An example of such a doubly periodic problem is pulsating flow through a regular structure or flow through structures which are regular in two dimensions. If the transition matrix $\mathbf{A}_k$ is independent of $k$ the problem is amenable to solution via Fourier series methods. However, even in this case the numerical method proposed here might still be advantageous.

In (1) the complete solution is obtained after $N-1$ recursions, provided an initial value is known. This can be obtained in a straightforward manner. Starting with $\mathbf{x}_0=\mathbf{x}_N$ and after $N$ substitutions from (1) it is found that the only unknown which remains is $\mathbf{x}_0$. e.g.

$$\begin{aligned}\mathbf{x}_0=\mathbf{x}_N&=\mathbf{A}_{N-1}\cdot\mathbf{x}_{N-1}+\mathbf{u}_{N-1}\\&=\mathbf{A}_{N-1}(\mathbf{A}_{N-2}\cdot\mathbf{x}_{N-2}+\mathbf{u}_{N-2})+\mathbf{u}_{N-1}\\&=\mathbf{A}_{N-1}(\mathbf{A}_{N-2}(\mathbf{A}_{N-3}(\ldots\mathbf{A}_1(\mathbf{A}_0\mathbf{x}_0+\mathbf{u}_0)+\mathbf{u}_1\ldots)+\mathbf{u}_{N-2})+\mathbf{u}_{N-1}\end{aligned} \tag{2}$$

In (2) the only unknown is $\mathbf{x}_0$ which is obtained from:

$$\left(\mathbf{I}-\prod_{k=0}^{N-1}\mathbf{A}_k\right)\cdot\mathbf{x}_0=\left(\sum_{k=0}^{N-2}\left\{\prod_{j=k+1}^{N-1}\mathbf{A}_j\right\}\cdot\mathbf{u}_k+\mathbf{u}_{N-1}\right) \tag{3}$$

In (3), equation (1) is effectively iterated through one cycle to determine the initial value. The complete solution is obtained by solving for $\mathbf{x}_0$ from (3) and then iterating (1) through one more cycle. It requires $2(N-1)$ products and one inversion. Note that the restriction to linear systems is required in order to solve $\mathbf{x}_0$ from (2), A must not be dependent on $\mathbf{x}_0$. When programming (3) advantage can be taken of the fact that the product expression occurs on both sides of the expression, by starting with the highest value of $k$ and counting down instead of up and storing at each step $k$ the partial sum and the partial product, the product expression can be successively constructed for every term in the sum.

Sometimes the discrete system takes the implicit form:

$$\mathbf{B}_{k+1}\cdot\mathbf{x}_{k+1}-\mathbf{C}_k\cdot\mathbf{x}_k=\mathbf{v}_{k+1} \tag{4}$$

with $\mathbf{B}_k$ and $\mathbf{C}_k$ square matrices. Equation (4) may be reduced to the form of (1) by the transformation:

$$\mathbf{A}_k = \mathbf{B}_{k+1}^{-1} \cdot \mathbf{C}_k, \qquad \mathbf{u}_k = \mathbf{B}_{k+1}^{-1} \cdot \mathbf{v}_{k+1} \tag{5}$$

The time dependent implicit system requires a matrix inversion at each step, however, for time invariant systems the inversion need only be performed once. Note that because the system is periodic, it is also possible to march backward in time with a similar solution scheme.

The solution for the initial value (3) enforces the cyclic condition $\mathbf{x}_0 = \mathbf{x}_N$ on the discrete system so that any discretization error in (1) at $k=N$ will equal the error at $k=0$. The cyclic condition is enforced even if the discrete system is weakly unstable, as long as the product $\prod_{k=0}^{N-1} \mathbf{A}_k$ remains finite. This inherent robustness and error propagation control are decided advantages of the cyclic assumption and a good reason for using cyclic models whenever feasible.

Of the discrete models (1) or (4) are derived from a discrete approximation of a continuous system, the sampling interval ill normally be contained in the matrices $\mathbf{A}_k$ or $\mathbf{B}_k$ and $\mathbf{C}_k$. Since it is assumed that these matrices are dependent on $k$, it is allowable to vary the sampling interval during the iteration, provided the time steps are selected initially. Equation (3) may also be used to find a good first guess of the initial value for a more accurate shooting method.

## APPLICATION

The form of the discrete system (1) or (4) is typical of systems which evolve with time, i.e. continuous systems which contain a first order time derivative term, so that $k+1$ in (1) and (4) refers to a discretization of the time axis. For this reason it is foreseen that the method will mostly find application in systems where the periodic independent variable is time. In this section we demonstrate the application to such a first order continuous system.

A highly simplified model of thermoflow in buildings is given in *Figure 1*. This single zone model is discussed in detail elsewhere[3]. Briefly, in *Figure 1* the resistances are: $R_0$=conductive shell resistance including exterior film coefficients, $R_a$=film resistance of the interior surfaces and $R_v$=effective ventilation resistance. The sources are: $T_{sa}$=averaged sol–air temperature of the external surfaces, $Q_r$=mean radiation on the interior surfaces, $Q_v$=interior air convective source and $T_0$=temperature of the ventilating air. The heat capacitance of the massive elements of the building is represented by the capacitor $C$. The dependent quantity of interest is the internal temperature represented by $T_i$. In practice some of the resistances will be time dependent since e.g. the ventilation rate varies with the hour of the day. In addition, modern ideas of passive climate control utilizes concepts such as night cooling and variable resistance. Night cooling is used in hot climates with cool night air, where the ventilation rate may be increased during cool

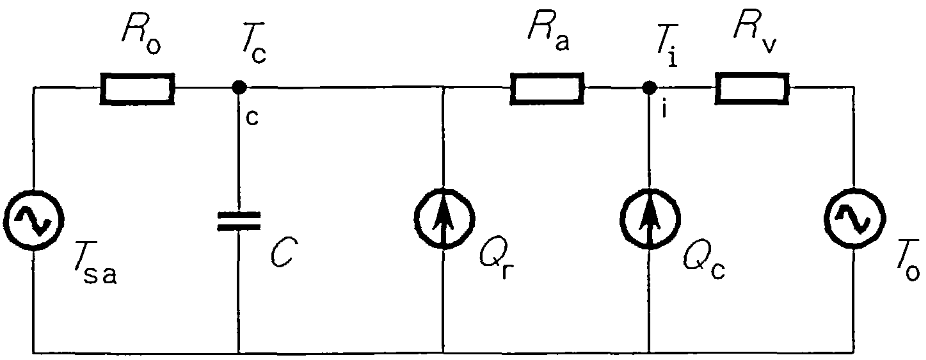


*Figure 1* Electrical analogue[3] for thermal analysis of buildings. $T_{sa}$=mean sol–air temperature; $T_c$=bulk structure temperature; $T_i$=interior air temperature; $T_0$=exterior air temperature; $R_0$=shell thermal resistance; $R_a$=mean interior surface film resistance; $R_v$=effective ventilation resistance, $C$ is the active heat storage capacitance of the structure

hours to absorb some excess heat from the structure. Variable shell resistance is used in cool climates to increase interior temperatures by increasing the resistance of the building envelope during cold hours and decreasing it during hot hours. Modelling of these techniques imply that the network of *Figure 1* must be solved under the assumption of variable resistances.

The governing equation of the network of *Figure 1* is:

$$\dot{x}(t)+\beta(t)\cdot x(t)=u(t) \tag{6}$$

with

$$x=C\cdot\left[\frac{T_i\cdot(R_v+R_a)-(T_0+R_v\cdot Q_c)\cdot R_a}{R_v}\right]$$

$$u=\frac{T_0+R_v\cdot Q_c}{R_a+R_v}+\frac{T_{sa}+R_0\cdot Q_r}{R_0}$$

and

$$\beta=\frac{R_a+R_v+R_0}{C\cdot R_0\cdot(R_a+R_v)}$$

A simple explicit method for solving (6) is:

$$x_{k+1}=(1-\Delta T\cdot\beta_{k+1})\cdot x_k+\Delta T\cdot u_k \tag{7}$$

where $\Delta T$ is the time step. If cyclic conditions are imposed i.e. $u_0=u_N$, $\beta_0=\beta_N$ and $x_0=x_N$ (3) is applicable and the initial value is found immediately.

$$x_0=\frac{\left(\sum_{k=0}^{N-2}\Delta T\cdot u_k\cdot\prod_{j=k+1}^{N-1}(1-\Delta T\cdot\beta_k)\right)+\Delta T\cdot u_{N-1}}{1-\prod_{k=0}^{N-1}(1-\Delta T\cdot\beta_k)} \tag{8}$$

The iteration (7) converges in practice if the step size is chosen so that $\Delta T\cdot\beta_k<2$ although the condition will not necessarily guarantee convergence in the time dependent case[4]. If the backward difference is used an unconditionally stable system is obtained. For buildings the time-constant $\tau=1/\beta$ lies in the range 2 to 200 h so that a time step of 15 min may be used.

Despite the extreme simplicity of the method it has been successfully applied as a design tool for a wide range of buildings of different construction. The simplicity of the model enables quick solutions on cheap computers even with complicated climate control systems. The diversity of buildings and the wide range of parameter values to which the method has been applied is indicated in *Tables 1* and *2*.

*Table 1* Numerical values of circuit parameters for some typical building zones. The value of the time-constant is given for ventilation rates of 0.1 and 30 ach (air changes per hour)

| | Thermal parameter | | | ach = 0.1 | | ach = 30 | |
|---|---|---|---|---|---|---|---|
| Building | $C$ [kJ/K] | $R_o$ [K/kW] | $R_a$ [K/kW] | $R_v$ [K/kW] | $\tau$ [h] | $R_v$ [K/kW] | $\tau$ [h] |
| Shed | 416521.70 | 0.0656 | 0.0650 | 9.934 | 7.54 | 0.0331 | 4.55 |
| Hut | 1822.77 | 33.8771 | 2.0330 | 1333.333 | 16.73 | 4.4444 | 2.75 |
| Factory | 1593885.15 | 0.0422 | 0.0060 | 0.335 | 16.62 | 0.0011 | 2.70 |
| Room | 3968.47 | 29.0070 | 2.0740 | 1333.333 | 31.30 | 4.4444 | 5.87 |
| Shop | 45398.16 | 2.7359 | 0.3150 | 69.231 | 33.19 | 0.2308 | 5.74 |
| Office | 61780.04 | 8.3667 | 1.4180 | 878.049 | 142.23 | 2.9268 | 49.08 |

*Table 2* Some construction data of the building zones in *Table 1*

| | Floor area [m²] | Shell area [m²] | Volume [m³] | Window area [m³] | Roof | Floor | Walls |
|---|---|---|---|---|---|---|---|
| Shed | 763 | 1290.8 | 3624 | 12.6 | exposed steel | concrete on ground | steel |
| Hut | 9 | 48.3 | 27 | 1.76 | exposed steel airspace fibreboard | concrete on ground | double brick |
| Factory | 7755 | 13292 | 107500 | 1335 | exposed steel glass wool | concrete on ground | steel glass wool |
| Room | 11 | 19.5 | 27 | 1.79 | exposed steel airspace glass wool gypsum | carpet concrete on ground | brick cavity brick |
| Shop | 102 | 185.7 | 520 | 24.92 | exposed slate glass wool airspace fibreboard | PVC concrete suspended | double brick |
| Office | 14 | 9.9 | 41 | 3.91 | not exposed concrete | PVC concrete | brick cavity brick |

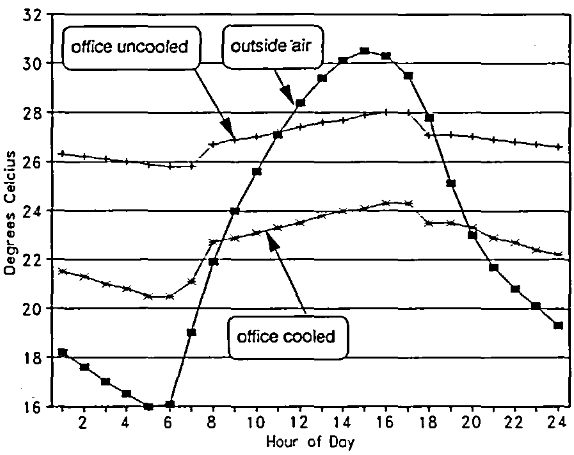


*Figure 2* Night cooling of a massive office. The trace marked 'uncooled' is with a ventilation-rate of 1 ach during the hours 17h00 to 09h00, and 2 ach during the hours 09h00 to 17h00. The other trace shows the effect of increasing the ventilation to 20 ach during the hours 20h00–06h00. The internal load during the hours 08h00 to 17h00 consists of 0.5 kW and 2 persons

Results obtained for a typical office is indicated in *Figure 2*. It shows the interior temperature obtained when the ventilation is increased during the cool night hours, together with the exterior air-temperature. The numerical values of the circuit elements for the office are supplied in *Table 1* with some physical details of the zone in *Table 2*. Note in *Figure 2* how the increased ventilation during the night succeeds in lowering the daytime temperature.

To obtain a practical evaluation of the accuracy of the method, the approximate solution can be compared with the exact solution in a special case. We take the case where $\beta(t)$ is constant everywhere, except at two points where the value jumps discontinuously i.e. $\beta(t)$ given by

$$\beta(t)=\begin{cases}\beta_0 & \text{when } 0\leqslant t<T_1\\ \beta_1 & \phantom{\text{when }} T_1\leqslant t<T\end{cases} \tag{9}$$

The exact solution for a constant $\beta$ and a sinusoidal input function given by $u=U\cdot(1+m\cdot\cos\omega[t+t_0])$ is from the Laplace transform:

$$x(t)=x(0)\cdot e^{-\beta t}+A(t) \tag{10}$$

where

$$A(t)=\frac{U}{\beta}\cdot\left[1-e^{-\beta t}+\frac{m\beta}{\sqrt{\beta^2+\omega^2}}\cdot\alpha(t)\right]$$

$$\alpha(t)=\cos(\omega[t+t_0]-\varphi)-\cos(\omega t_0-\varphi)\cdot e^{-\beta t}$$

and $x(0)$ is the initial value, $\tan\varphi=\omega/\beta$. Next, apply this solution to the intervals in (9) and set $x(0)=x(T)$, and $x(T_1)$ continuous. The solution for $x(0)=x_0$ is:

$$x_0=\frac{A_1+A_0\cdot e^{-\beta_1(T-T_1)}}{1-e^{-\beta_0 T_1-\beta_1(T-T_1)}} \tag{11}$$

with

$$A_0=A(T_1) \qquad \text{and} \qquad A_1=A(T_1-T_1).$$

The subscripts 0 and 1 of $A$ and $\beta$ in (11) refer to the first and second intervals in (9) respectively. *Figure 3* shows the error between analytic solution (10), (11), and approximate numerical solution (7), (8), for a building with relatively short time-constant. The building (an agricultural shed) has a time-constant of 7.5 hours (see *Tables 1* and *2*) with closed windows. This is a very short thermal time-constant and a practical sampling rate would be 15 min, but to show the robustness of the method, a sampling periof of 1 h is used in the calculation. The ventilation rate jumps from 0.1 to 30 ach resulting in a time-constant jump from 7.5 to 4.5 h, the jump occurring at

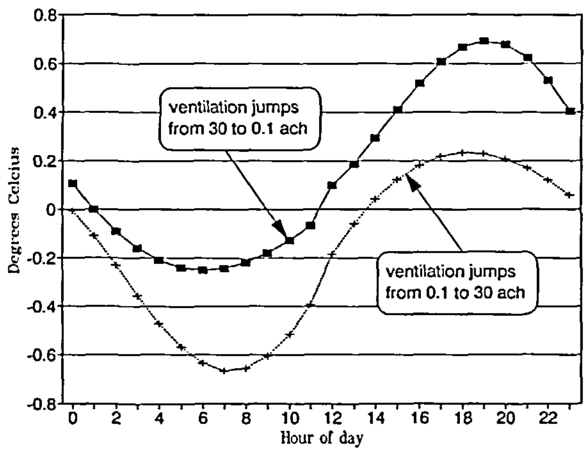


*Figure 3* Difference between analytically derived exact prediction of interior temperature and numerical algorithm for a sudden jump in ventilation rate and sinusoidal forcing functions. The sampling period is 1 h. Building: agricultural shed, time-constant $\tau=4.5$ h with ventilation rate 30 ach. The upper and lower traces show the error when the ventilation-rate jumps from 0.1 to 30 ach, and from 30 to 0.1 ach respectively, the jump occurring at the 11th hour

$T_1 = 11$ h. The forcing functions used for the calculation are:

$$T_{sa} = 20 + 10 \cdot \cos(2\pi/24 \cdot t)°C$$

$$T_0 = 20 + 5 \cdot \cos(2\pi/24 \cdot t)°C$$

and

$$Q_c = Q_r = 0\,\text{kW}$$

The Figure shows the error obtained by a sudden increase in the number of air changes as well as a sudden decrease of similar strength. In this worst case, $\beta_{max} \cdot \Delta T = 1/4.5$, the temperature error is less than 1°C. The error is decreased to insignificant levels by decreasing the sampling period to 15 min, with linear interpolation between sampling points. The calculations were repreated for a building with a longer time-constant (office block), where the time constant jumped from 142 to 49 h when the ventilation rate jumped from 0.1 to 30 ach. The error between the analytic and approximate solutions in this case, $\Delta T = 1$ h, $\beta_{max} \cdot \Delta T = 1/49$, was less than 0.1°C.

## EFFICIENCY OF THE METHOD

It is difficult to give a general discussion of the relative efficiency of the method since it varies considerably from application to application. In buildings, thermal time-constants may be as long as 200 h or more. Many existing programs assume an arbitrary initial value and integrate until the cyclic condition is satisfied. A rule of thumb is that the transient response will be negligible after 5 time constants, that is after 1000 h or approximately 42 cycles. An efficient shooting method will require a minimum of three cycles. In contrast, the solution given here requires effectively two cycles to obtain the full solution.

An alternative procedure would be to solve the system simultaneously for every time step through a full cycle. For the building thermal analysis example above this will require the simultaneous solution of 96 equations if the single order approximation outlined above is followed, with a time step of 15 min. Obviously the system of equations would be very sparse and could be solved efficiently provided the minimum bandwidth representation could be found[5]. In this case, once the minimum bandwidth representation has been found, the number of computations required would be proportional to $N \cdot w^2$ where $N$ is the number of equations and $w$ the minimum bandwidth. In general the solution obtained through (3) also requires computations proportional to $N$ so that it appears to be of the same order of efficiency, however, it is not required to find the minimum bandwidth representation first. In addition, it only requires computer storage of the state for two values of the index $k$ in (3), while the sparse matrix method requires storage for all values of $k$.

## CONCLUSION

Cyclic systems have many applications in heat transfer and fluid flow. They can be efficiently solved with the method presented here if the system is linear or can be linearized. The method caters for time dependent systems as well as variable step-size systems. The efficiency of the solution and the inherent error control of the cyclic system can be exploited by specifying cyclic boundary conditions even if the physical system is only approximately cyclic.

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